From frequencies to masses: an action approach to normalized solutions

Stable and unstable Dynamics in PDEs and Hamiltonian Systems
BIRS–CMO, Oaxaca, Mexico

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The nonlinear Schrödinger *evolution* equation

We consider the problem

$$\begin{cases} i\partial_t\psi = -\Delta\psi - |\psi|^{p-2}\psi, & (t,x) \in [0, T[\times \Omega, \\ \psi(t,x) = 0, & (t,x) \in [0, T[\times \partial\Omega, \\ \psi(0,x) = \psi_0(x), & \psi_0 : \Omega \to \mathbb{C}, x \in \Omega \end{cases} \qquad (\mathrm{NLS}_{\mathrm{evol}})$$

with $\psi : [0, T[\times \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{R}^N$ a *bounded* domain, $N \geq 1$, and $p > 2$.

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with $\psi : [0,T[\times\Omega \to \mathbb{C}$, $\Omega \subset \mathbb{R}^N$ a *bounded* domain, $N \geq 1$, and $p > 2$.

At least formally, the L^2 *norm* (the *mass*) $\|\psi(t,\cdot)\|^2_{L^2(\Omega)}$ and the *energy*

$$E\Big(\psi(t,\cdot)\Big) := \frac{1}{2}\int_\Omega |\nabla_x\psi|^2\,\mathrm{d}x - \frac{1}{p}\int_\Omega |\psi|^p\,\mathrm{d}x$$

are *preserved* during the evolution.

Solitary wave solutions

Opposed to blow-up: *solitary waves* of the form

$$\psi(t,x) = \mathrm{e}^{\boldsymbol{i}\lambda t} u(x)$$

where $u \in H_0^1(\Omega; \mathbb{R}) = H_0^1(\Omega)$ is a solution of

$$-\Delta u + \lambda u = |u|^{p-2} u. \tag{NLS}$$

Some vocabulary:

- $\lambda \in \mathbb{R}$ is the *frequency* of the solitary wave;
- $\|u\|_{L^2(\Omega)}^2 = \|\psi(t,\cdot)\|_{L^2(\Omega)}^2$ is its *mass*.

Two problems, two functionals

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Given $\lambda \in \mathbb{R}$, how to find a nonzero stationary wave of frequency λ?

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The *energy functional* and, given $\lambda \in \mathbb{R}$, the *action functional*:

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x,$$

$$J_\lambda(u) := E(u) + \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x.$$

Variational formulations

Proposition

Given $2 < p < 2^*$ and $\lambda \in \mathbb{R}$, solutions of frequency λ correspond to critical points of J_λ on $H_0^1(\Omega)$.

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Given $2 < p < 2^*$ and $\mu > 0$, normalized solutions of mass μ correspond to constrained critical points of E on the L^2-sphere

$$\mathcal{M}_\mu := \left\{ u \in H_0^1(\Omega) \mid \|u\|_{L^2(\Omega)}^2 = \mu \right\}.$$

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In the case of normalized solutions, the parameter λ in the PDE will appear as a Lagrange multiplier associated with the constraint.

Lower boundedness of the energy functional

Proposition

Let $2 < p < 2^*$ *and* $\mu > 0$. *Then:*

- *if* $2 < p < 2 + 4/N$,
$$\inf_{\mathcal{M}_\mu} E > -\infty;$$
- *if* $2 + 4/N < p < 2^*$,
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The boundedness follows from the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p(\Omega)} \leq C(p)\,\|u\|_{L^2(\Omega)}^{1-s}\,\|\nabla u\|_{L^2(\Omega)}^{s}, \quad s := \frac{(p-2)N}{2p}.$$

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and the unboundedness by considering the limit $t \to +\infty$ for a family $t^{N/2}\psi(tx)$, with constant L^2-norms, obtained by scaling a fixed profile.

A classic result and two questions

Proposition

When $\mu > 0$ and $2 < p < 2 + 4/N$, then minimizers for E on $\mathcal{M}_\mu$ exist, have a constant sign and are normalized solutions of (NLS). They are called energy ground states.

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How to find sign-changing normalized solutions?

Answers: given by the results of the talk!

The Nehari manifold

J_λ is unbounded from below on $H_0^1(\Omega)$: $J_\lambda(tu) \to -\infty$ as $t \to +\infty$, for $u \neq 0$. A common strategy is the *Nehari manifold*

$$\mathcal{N}_\lambda := \left\{ u \in H_0^1(\Omega) \setminus \{0\} \mid J_\lambda'(u)[u] = 0 \right\}$$
$$= \left\{ u \neq 0 \mid \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 = \|u\|_{L^p(\Omega)}^p \right\}.$$

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If $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{L^p(\Omega)}^p.$$

In particular, J_λ is bounded from below on $\mathcal{N}_\lambda$.

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Proposition

Given $\lambda > -\lambda_1(\Omega)$ and $2 < p < 2^*$, then minimizers for J_λ on $\mathcal{N}_\lambda$ exist, have a constant sign and are solutions of (NLS) having frequency λ. They are called action ground states.

Nodal action ground states

One defines the nodal Nehari set by

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Theorem (Castro, Cossio, Neuberger 1997; Bartsch-Weth 2003)

Given $\lambda > -\lambda_2(\Omega)$ and $2 < p < 2^*$, then minimizers for J_λ on $\mathcal{N}_\lambda^{nod}$ exist, have two nodal zones and are solutions of (NLS) having frequency λ. They are called nodal action ground states.

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Remark: I will use the terms "sign-changing" and "nodal" interchangeably, as the contrary of "one-signed".

Comparison of the two settings so far

Abbreviation: "ground state" $\rightarrow$ GS

	$2 < p < 2 + 4/N$	$2 + 4/N < p < 2^*$
Positive solution	Energy GS	?
Sign-changing solution	?	?

The fixed mass μ case

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Positive solution	Action GS	Action GS
Sign-changing solution	Nodal action GS	Nodal action GS

The fixed action λ case

Sign-changing normalized solutions

There are few works about sign-changing normalized solutions — let us mention a recent preprint of Jeanjean and Song, using gradient flow techniques.

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In the literature, there is no equivalent of the nodal Nehari set for normalized solutions and it is in fact very unclear if such a nice "codimension two constraint" does exist for this problem.

Positive normalized solutions, L^2-supercritical

Since Jeanjean's pioneering work in the late 90s, many studies in the regime $2 + 4/N < p < 2^*$. Two main difficulties:

- the energy is unbounded from below on the constraint $\rightarrow$ "mountain pass" solutions;

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On bounded domains: Noris-Tavares-Verzini 2014 (*on the ball*, via uniqueness of the positive solution); Pierotti-Verzini 2017, Pierotti-Verzini-Yu 2025 (general domains, solutions *with a control on the Morse index*; the latter needs star-shapedness for all masses).

Action versus energy ground states

Both notions have a long history, but a systematic comparison was not made until recently: *if energy ground states exist, they are necessarily action ground states for the corresponding λ — the converse being false!* (Dovetta-Serra-Tilli 2023.)

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Theorem (Dovetta-Serra-Tilli 2023)

Let $2 < p < 2 + 4/N$ and Ω be bounded. Setting

$$\mathcal{E}(\mu) := \inf_{u \in \mathcal{M}_\mu} E(u), \qquad \mathcal{J}(\lambda) := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u),$$

the function $-\mathcal{E}(2\,\cdot)$ is the Legendre-Fenchel transform of $\mathcal{J}$:

$$-\mathcal{E}(2\mu) = \sup_{\lambda \in \mathbb{R}} \Big(\lambda\mu - \mathcal{J}(\lambda) \Big).$$

Main message

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More precisely:

- using such a "convex duality argument" from the action ground states when $2 + 4/N < p < 2^*$ will *also* produce normalized solutions;
- doing so from the nodal action GS will produce sign-changing normalized solutions, which is new for all $2 < p < 2^*$.

Our result (for positive solutions)

Theorem (De Coster-Dovetta-G.-Serra 2025)

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and, for every $2 < p < 2^*$, let

$$M_p := \left\{ \|u\|^2_{L^2(\Omega)} \;\middle|\; u \in \mathcal{N}_\lambda \text{ and } J_\lambda(u) = \mathcal{J}(\lambda) \text{ for some } \lambda \in \mathbb{R} \right\}$$

be the set of masses of all action ground states. Then,

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(i) *if $2 < p < 2 + 4/N$, then $M_p = (0, +\infty)$;*

(ii) *if $p = 2 + 4/N$, then there exists $0 < \mu_p < +\infty$ such that either $M_p = (0, \mu_p)$ or $M_p = (0, \mu_p]$;*

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(iii) *if $2 + 4/N < p < 2^*$, then there exists $0 < \mu_p < +\infty$ such that $M_p = (0, \mu_p]$.*

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In fact, this map is not even well-defined as action GS might not be unique.

Remark: such a C^1 branch $\lambda \mapsto u_\lambda$ is exactly what stability theory "à la Grillakis-Shatah-Strauss" assumes from the outset.

Properties of the action level map $\lambda \mapsto \mathcal{J}(\lambda)$

Proposition (Dovetta-Serra-Tilli 2023)

Let $2 < p < 2^*$. Then:

(i) *For every $\lambda \leq -\lambda_1(\Omega)$, $\mathcal{J}(\lambda) = 0$ and action ground states in $\mathcal{N}_\lambda$ do not exist.*

(ii) *For every $\lambda > -\lambda_1(\Omega)$, $\mathcal{J}(\lambda) > 0$ and action ground states in $\mathcal{N}_\lambda$ exist.*

(iii) *The function $\mathcal{J} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and increasing on $[-\lambda_1(\Omega), +\infty)$.*

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(iii) *The function* $\mathcal{J} : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz continuous and increasing on* $[-\lambda_1(\Omega), +\infty)$.

Moreover, "derivatives of $\mathcal{J}$ give L^2-masses of action ground states" (to be precised) — this too is theirs.

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But still a good heuristic :-)

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Recalling the definition of J_λ, we have

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Of course, we have that $\mathcal{J}(\lambda) = J_\lambda(u_\lambda)$ for a **varying** action GS u_λ (they must be in different Nehari manifolds!). It just so happens that the action GS change "little enough" that the leading term is the same than if the minimizer was fixed, which is extremely convenient.

A correct version of the heuristic argument

Proposition

Let $2 < p < 2^*$, $\lambda > -\lambda_1(\Omega)$ *and define*

$$Q_p(\lambda) := \left\{ \|u\|^2_{L^2(\Omega)} \mid u \in \mathcal{N}_\lambda \text{ and } J_\lambda(u) = \mathcal{J}(\lambda) \right\}$$

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be the set of masses of action ground states. Then, we have

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{\mathcal{J}(\lambda+\varepsilon) - \mathcal{J}(\lambda)}{\varepsilon} &= \frac{1}{2} \inf Q_p(\lambda) \\ &\leq \frac{1}{2} \sup Q_p(\lambda) = \lim_{\varepsilon \to 0^-} \frac{\mathcal{J}(\lambda+\varepsilon) - \mathcal{J}(\lambda)}{\varepsilon}, \end{aligned}$$

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Moreover, for every λ *outside an at most countable set, all action ground states have the same mass (i.e.,* $Q_p(\lambda)$ *is a singleton).*

A miracle

Proposition (Key proposition)

Let $\mu > 0$ and $2 < p < 2^*$. Assume that $\lambda_* > -\lambda_1(\Omega)$ is a local minima of the map $f_\mu : [-\lambda_1(\Omega), +\infty) \to \mathbb{R}$ defined by

$$f_\mu(\lambda) := \mathcal{J}(\lambda) - \tfrac{1}{2}\mu\lambda.$$

Then, $\mathcal{J}$ is differentiable for $\lambda = \lambda_*$ and one has that $\mathcal{J}'(\lambda_*) = \frac{1}{2}\mu$, so that all action ground states with $\lambda = \lambda_*$ have mass μ.

Proof of the key proposition

Proof.

At a minimum point, one must have

$$\limsup_{\varepsilon\to 0^-} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon} \leq 0 \leq \liminf_{\varepsilon\to 0^+} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon},$$

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namely

$$\limsup_{\varepsilon\to 0^-} \frac{\mathcal{J}(\lambda_* + \varepsilon) - \mathcal{J}(\lambda_*)}{\varepsilon} \le \tfrac{1}{2}\mu \le \liminf_{\varepsilon\to 0^+} \frac{\mathcal{J}(\lambda_* + \varepsilon) - \mathcal{J}(\lambda_*)}{\varepsilon}.$$

□

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Proof.

At a minimum point, one must have

$$\limsup_{\varepsilon\to 0^-} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon} \leq 0 \leq \liminf_{\varepsilon\to 0^+} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon},$$

namely

$$\limsup_{\varepsilon\to 0^-} \frac{\mathcal{J}(\lambda_* + \varepsilon) - \mathcal{J}(\lambda_*)}{\varepsilon} \leq \tfrac{1}{2}\mu \leq \liminf_{\varepsilon\to 0^+} \frac{\mathcal{J}(\lambda_* + \varepsilon) - \mathcal{J}(\lambda_*)}{\varepsilon}.$$

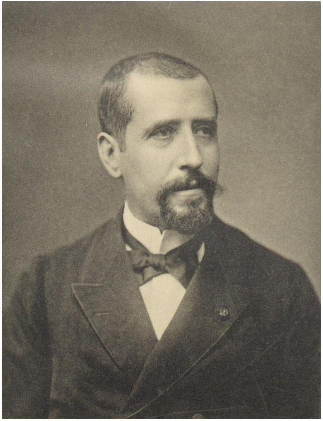
But we just saw that the reverse inequality holds! $\square$

Interlude: Darboux's theorem for derivatives

Somehow, we just proved a "Darboux-type" theorem for $\mathcal{J}'$ (even though $\mathcal{J}'$ is not pointwise well-defined). As a comparison, here is Darboux's original theorem.

Theorem (Darboux 1875)

Let $f : I \to \mathbb{R}$ be differentiable, where I is an interval. Then, $f'(I)$ is an interval.



Jean-Gaston Darboux (1842–1917).

Image from Wikimedia Commons.

Asymptotic behavior of $\mathcal{J}$

Proposition ($\lambda \to -\lambda_1(\Omega)$)

For every $2 < p < 2^*$, *there exist* $C_1, C_2 > 0$ *such that, for every* $\lambda \geq -\lambda_1(\Omega)$,

$$C_2 \min\left(1, \frac{\lambda+\lambda_1(\Omega)}{\lambda_1(\Omega)}\right)^{\frac{p}{p-2}} \leq \mathcal{J}(\lambda) \leq C_1(\lambda + \lambda_1(\Omega))^{\frac{p}{p-2}}.$$

In particular $\mathcal{J}(\lambda)/(\lambda + \lambda_1(\Omega)) \to 0$ *as* $\lambda \to -\lambda_1(\Omega)$, $\lambda > -\lambda_1(\Omega)$.

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In particular $\mathcal{J}(\lambda)/(\lambda + \lambda_1(\Omega)) \to 0$ *as* $\lambda \to -\lambda_1(\Omega)$, $\lambda > -\lambda_1(\Omega)$.

Proposition ($\lambda \to +\infty$)

$$\lim_{\lambda \to +\infty} \frac{\mathcal{J}(\lambda)}{\lambda} = \begin{cases} +\infty & \text{if } 2 < p < 2 + 4/N, \\ 0 & \text{if } 2 + 4/N < p < 2^*. \end{cases}$$

Putting it all together

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This proves our announced results for positive solutions.

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ceases to be a norm;
- we have to rely on Bartsch-Weth's (non-trivial!) result to obtain existence of nodal action ground states when $-\lambda_2(\Omega) < \lambda \leq -\lambda_1(\Omega)$;
- the claims can be adapted quite naturally to the nodal setting and proved in analogous ways, up to the above remarks.

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We may however be interested in least energy normalized (nodal) solutions, namely solutions having least energy among all (nodal) solutions.

On $\mathbb{R}^N$, least energy normalized solutions are known to exist for every mass when $p \neq 2+4/N$.

A counterintuitive fact when $p = 2 + 4/N$

One can show that least energy solutions (resp. least energy nodal solutions) exist for all $\mu \in (0, \mu_N)$ (there are possibly more), resp. for all $\mu \in (0, 2\mu_N)$, where μ_N is the mass of the corresponding soliton on $\mathbb{R}^N$.

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In the critical and supercritical cases...

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This strikingly shows that not all properties of energy ground states transfer to least energy normalized solutions.

Taking one step back: the role of the spectrum

Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the Dirichlet eigenvalues of $-\Delta$ on Ω, with eigenfunctions φ_j. Everything in the action approach was driven by

$$q_\lambda(u) := \|\nabla u\|^2_{L^2(\Omega)} + \lambda\|u\|^2_{L^2(\Omega)}, \qquad q_\lambda(\varphi_j) = (\lambda_j + \lambda)\|\varphi_j\|^2_{L^2(\Omega)}.$$

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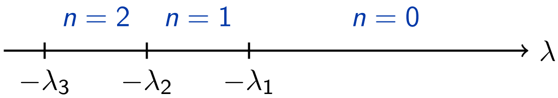

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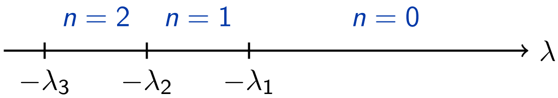

The case $n = 0$ is exactly $\lambda > -\lambda_1$: the setting of the Nehari manifold.

The Nehari-Pankov set

Let $E_n := \mathrm{span}\{\varphi_1, \dots, \varphi_n\}$, with n the number of negative directions of q_λ. The *Nehari-Pankov set* is

$$\mathcal{NP}_\lambda := \left\{ u \in H_0^1(\Omega) \setminus E_n \;\middle|\; J'_\lambda(u)[u] = 0 \text{ and } J'_\lambda(u)[v] = 0 \ \ \forall v \in E_n \right\}.$$

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One sets again $c_\lambda := \inf_{\mathcal{NP}_\lambda} J_\lambda$ (for $n = 0$, this is $\mathcal{J}(\lambda)$), which now has a *saddle point* nature and admits an inf-sup characterization.

A second step back: which operator?

So far we took *one* step back, along a single axis: with $A = -\Delta$ on Ω **fixed**, the constraint follows the spectrum as λ decreases, $\mathcal{N}_\lambda \rightsquigarrow \mathcal{NP}_\lambda$.

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So let $A : D(A) \subset H \to H$ be self-adjoint, non-negative, with compact resolvent (H a separable real Hilbert space), let $X := D(A^{1/2})$ and let $I \in C^1(X, \mathbb{R})$ be superlinear. We consider

$$Au + \lambda u = I'(u), \qquad J_\lambda(u) = \frac{1}{2}\left(\|A^{1/2}u\|_H^2 + \lambda\|u\|_H^2\right) - I(u).$$

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Then q_λ, the spectral gaps, the sets $\mathcal{NP}_\lambda$ and the levels c_λ are defined *verbatim* as before.

Connectedness of the masses, one level up

Theorem (G.-Weth 2026)

Fix $n \geq 1$, let λ range over the spectral gap $(-\lambda_{n+1}, -\lambda_n)$ and let

$$M_n := \left\{ \|u\|_H^2 \,\middle|\, u \in \mathcal{NP}_\lambda,\ J_\lambda(u) = c_\lambda \text{ for some such } \lambda \right\}$$

be the set of masses of Nehari-Pankov ground states. Then

$$(0, g_n) \subseteq M_n \subseteq (0, g_n] \qquad \text{for some } g_n > 0.$$

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*But the proof is genuinely harder: connectedness is **not** a consequence of the continuity of $\lambda \mapsto c_\lambda$ alone — it uses the saddle point nature of $\mathcal{NP}_\lambda$.*

From connectedness to multiplicity

Theorem (G.-Weth 2026)

Let $k \geq 1$ and let $I_1, \ldots, I_k$ be disjoint spectral gaps of A. Then there exists $\mu_k > 0$ such that, for every $\mu \in (0, \mu_k)$, there exist at least k distinct normalized solutions of mass μ.

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But this only gives ***small*** *masses.* For *infinitely many* solutions at *every* mass, one needs $g_n \not\to 0$ as $n \to \infty$: that is, effective *lower* bounds on g_n.

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Instead, one exploits the *length* of the spectral gap $(-\lambda_{n+1}, -\lambda_n)$, together with:

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Note: nothing here refers to a mass-critical exponent. The method is insensitive to it — which is precisely what makes the mass-supercritical case accessible.

Application 1: NLS on compact metric graphs

Let $\mathcal{G}$ be a compact metric graph and A *any* self-adjoint realisation of the Laplacian on $\mathcal{G}$ (the second derivative on each edge). Assume only that f is *superlinear* in a natural sense ($s \mapsto f(s)/|s|$ strictly increasing, from 0 at the origin to $\pm\infty$ at $\pm\infty$) and *approximately homogeneous* (a mild condition, automatic when f is odd).

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No parity of f*, no growth restriction. New even for* ***one*** *non-constant solution at arbitrary mass.*

So no $\mathbb{Z}_2$-symmetry, hence *no Lusternik-Schnirelmann theory*: the multiplicity comes from the **spectrum** — one solution per large gap.

Why it works: Weyl's law and an ODE argument

- Weyl's law gives *arbitrarily large* spectral gaps $(-\lambda_{n+1}, -\lambda_n)$ along a subsequence — valid for *every* self-adjoint realisation of the Laplacian on a compact graph;

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- for λ near the right end $-\lambda_n$ of a large gap, classical spectral theory forces solutions to have a *large L^∞-norm*;
- an ODE (phase plane) argument turns this into an "L^∞ to L^2" estimate — giving $g_n \not\to 0$, hence *every* mass.

Application 2: polyharmonic equations

Theorem (G.-Weth 2026)

Let $\Omega \subset \mathbb{R}^N$ be smooth and bounded, and consider

$$(-\Delta)^m u + \lambda u = f(x,u) \quad \text{in } \Omega,$$

with Dirichlet or Navier boundary conditions. For every integer $k \geq 1$ there exists $\mu_k > 0$ such that, for every $\mu \in (0, \mu_k)$, both problems admit k distinct solutions of mass $\|u\|^2_{L^2(\Omega)} = \mu$. If $m = 1$, these solutions are sign-changing.

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New even for $m = 1$, and no oddness of f is required.

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Let $\Omega \subset \mathbb{R}^N$ be smooth and bounded, and consider

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with Dirichlet or Navier boundary conditions. For every integer $k \geq 1$ there exists $\mu_k > 0$ such that, for every $\mu \in (0, \mu_k)$, both problems admit k distinct solutions of mass $\|u\|^2_{L^2(\Omega)} = \mu$. If $m = 1$, these solutions are sign-changing.

New even for $m = 1$, and no oddness of f is required.

*Only **small** masses — weaker than the graph case. We conjecture that for $N = 2$, $m > 1$, it extends to all $\mu > 0$.*

Application 3: the biharmonic equation on the 2-torus

Theorem (G.-Weth 2026)

Let $\mathbb{T}^2 = S^1 \times S^1$*, let* $p > 2$ *and let* $r \in L^\infty(\mathbb{T}^2)$ *with* $\operatorname{ess\,inf} r > 0$*. For* ***every*** $\mu > 0$,

$$\Delta^2 u + \lambda u = r(x)|u|^{p-2}u \quad \text{in } \mathbb{T}^2$$

admits ***infinitely many*** *normalized sign-changing solutions of mass* $\|u\|^2_{L^2(\mathbb{T}^2)} = \mu$*, along a sequence* $\lambda \to -\infty$*.*

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The eigenvalues of Δ^2 on $\mathbb{T}^2$ are $(j_1^2 + j_2^2)^2$: the gaps are estimated with a classical bound from *analytic number theory* on the number of representations as a sum of two squares.

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The eigenvalues of Δ^2 on $\mathbb{T}^2$ are $(j_1^2 + j_2^2)^2$: the gaps are estimated with a classical bound from *analytic number theory* on the number of representations as a sum of two squares.

For $p > 6$ (mass-supercritical), even ***one*** *normalized solution is new.*

A word on (in)stability

Direct minimization produces **minimizers** of E on the mass constraint — the classically *orbitally stable* solutions (Cazenave-Lions 1982).

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The action approach reaches solutions that are **likely unstable** — precisely what direct minimization cannot produce.

¡Muchas gracias!

References

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